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GEOSTROPHIC VORTICES ON A CIRCLE OF LATITUTDE IN A CAP ON A ROTATING SPHERE

A. S. Peters

New York University

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This report contains an analysis of the linear stability of the motion of n+1 geostrophic vortices confined in a polar cap on a rotating sphere. It is supposed that initially one vortex is at the north pole while the others are symmetrically arranged on a circle of latitude; and that the motion is subject to a certain

20. Abstract

condition along the latitude circle which bounds the cap. The development stems from the demonstration that a concentrated geostrophic vortex in a rotating spherical layer is characterized by a singular spherical harmonic of degree ν and order zero. This function has a role analogous to that of the Bessel function which characterizes a rectilinear geostrophic vortex in a rotating planar layer.

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1. Introduction

A thin spherical layer of an incompressible, inviscid fluid which is held on the surface S of a rotating ball by gravitation can be taken for some purposes as an approximation to the Earth's atmosphere. An investigation of the two dimensional vortical motion in such a layer should be useful for the understanding of certain observed meteorological phenomena. For example, it appears that the analysis of large scale closed isobaric systems can be based on a knowledge of the paths of concentrated vortices.

This report presents a study of the vortical motion which is due to the existence of concentrated vortices (normal to S) which are confined to a polar cap and subject to a boundary condition along a circle of latitude.

We assume that the departure of the free outer surface of the layer from an equilibrium position is small; and that the tangential acceleration is negligible compared with the Coriolis force. We also assume that the variation of the Coriolis force with latitude can be neglected In other words, we study geostrophic vortices on a sphere as contrasted with geostrophic vortices on a plane. The latter have been discussed by several authors in connection with the motion in a rotating planar tangential layer as an approximation to the motion in a thin layer of fluid covering a rotating ball. References can be found in the paper by Morikawa, [1].

The main result is the system of nonlinear equations for the paths of the vortices. These are used to study the linear stability of the motion of n vortices which are symmetrically arranged

along a circle of latitude in a cap with or without a polar vortex. One of the boundary conditions imposed requires the velocity to the north to be zero along the circle which bounds the cap; and the other requires the velocity to the east to be constant along the boundary. The investigation is restricted to analytical results which include a formula for linear stability for the case in which the polar vortex is either held fixed or else has zero strength. Numerical computations, which are necessary for decisions about the linear and nonlinear stability of given configurations of vortices, will be discussed in another report. See Leiva, [2].

2. Equations of Motion

Let ρ denote the distance of a point from the center of a ball E of large radius a which rotates with constant angular velocity ω about a polar axis. Let ϕ and θ denote respectively the longitude and the colatitude of a point on the rotating spherical surface S of E. Let $\rho = a$ and $\rho = a + h(\phi, \theta, t)$ represent two surfaces which contain an incompressible, inviscid fluid which is gravitationally attracted by E. Suppose h is small compared with a.

The velocity of a fluid particle relative to S is defined by the components

$$u = (\rho \sin \theta) \frac{d\phi}{dt} = \text{tangential component toward the east,}$$

$$v = -\rho \frac{d\theta}{dt} = \text{tangential component toward the north,}$$

$$w = \frac{d\rho}{dt} = \text{radial component.}$$

If the only body force acting is that due to the gravitational potential G of E, then the basic hydrodynamical equations which define the motion of the fluid relative to S are the continuity equation

$$\rho \frac{\partial w}{\partial \rho} + 2w + \frac{1}{\sin \theta} \left[\frac{\partial u}{\partial \phi} - \frac{\partial (v \sin \theta)}{\partial \theta}\right] = 0$$

and the momentum equations

$$\frac{du}{dt} - \frac{uv \cot \theta}{\rho} + \frac{uw}{\rho} + 2\omega w \sin \theta - 2\omega v \cos \theta = - \frac{1}{\delta_o \rho \sin \theta} \frac{\partial p_1}{\partial \phi} ;$$

$$\frac{dv}{dt} + \frac{vw}{\rho} + \frac{u^2 \cot\theta}{\rho} + 2\omega u \cos\theta = \frac{1}{\delta_o\rho}\frac{\partial p_1}{\partial\theta} ;$$

$$\frac{dw}{dt} - \frac{v^2}{\rho} - \frac{u^2}{\rho} - 2\omega u \sin\theta \qquad = -\frac{1}{\delta_o}\frac{\partial p_1}{\partial\rho} .$$

In these equations the differential operator with respect to the time means

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{u}{\rho \sin\theta}\frac{\partial}{\partial\phi} - \frac{v}{\rho}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial\rho} .$$

The symbol δ_o denotes the constant density of the fluid; and p stands for the pressure in

$$p_1 = p + \delta_o G - \frac{\delta_o\rho^2\omega^2 \sin^2\theta}{2}$$

which can be referred to as a modified pressure.

Since h is supposed to be small compared with the large radius a of E; and since

$$w(\phi, \theta, a, t) = 0$$

let us neglect the radial velocity and the radial variation of u and v. Let us assume that $G = g\rho$ where g is a constant; and that the centrifugal effects manifested by the partial derivatives of $\frac{\delta_o\rho^2\omega^2 \sin^2\theta}{2}$ can be ignored. Let us also assume that the motion is such that the nonlinear terms in the tangential momentum equations can be neglected; and that the radial momentum equation can be replaced with the hydrostatic law

$$p(\phi, \theta, \rho, t) = g\delta_o(h + a - \rho)$$

which satisfies the condition that the pressure is zero at the free surface $\rho = a+h$.

Under the above assumption and with the notation

$$\tilde{u}(\phi,\theta,t) = u(\phi,\theta,a,t) ,$$

$$\tilde{v}(\phi,\theta,t) = v(\phi,\theta,a,t) ;$$

an approximation to the motion is determined by the equations

$$(2.1) \qquad \frac{\partial w}{\partial \rho} + \frac{1}{a \sin \theta}\left[\frac{\partial \tilde{u}}{\partial \phi} - \frac{\partial(\tilde{v} \sin \theta)}{\partial \theta}\right] = 0 ;$$

$$(2.2) \qquad \frac{\partial \tilde{u}}{\partial t} - 2\omega\tilde{v} \cos \theta = - \frac{g}{a \sin \theta} \frac{\partial h}{\partial \phi} ;$$

$$(2.3) \qquad \frac{\partial \tilde{v}}{\partial t} + 2\omega\tilde{u} \cos \theta = \frac{g}{a} \frac{\partial h}{\partial \theta} .$$

An integration of (2.1) gives

$$w(\phi,\theta,a+h,t) = \frac{h}{a \sin \theta}\left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi}\right] .$$

The kinematic condition at the free surface is

$$w(\phi,\theta,a+h,t) = \frac{dh}{dt}$$

and hence

$$(2.4) \qquad \frac{dh}{dt} = \frac{h}{a \sin \theta}\left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi}\right] .$$

We suppose now that $h(\phi,\theta,t)$ is always close to the constant value h_o. Then if we introduce

$$\eta(\phi,\theta,t) = \frac{h - h_o}{h_o}$$

a linearization of (2.4) yields

$$\eta_t = \frac{1}{a \sin\theta}\left[\frac{\partial(\tilde{v}\sin\theta)}{\partial\theta} - \frac{\partial\tilde{u}}{\partial\phi}\right] \tag{2.5}$$

while the momentum equations become

$$\frac{\partial\tilde{u}}{\partial t} - 2\omega\tilde{v}\cos\theta = -\frac{gh_o}{a\sin\theta}\frac{\partial\eta}{\partial\phi}\,; \tag{2.6}$$

$$\frac{\partial\tilde{v}}{\partial t} + 2\omega\tilde{u}\cos\theta = \frac{gh_o}{a}\frac{\partial\eta}{\partial\theta}\,. \tag{2.7}$$

Unless stated otherwise, we assume in what follows that ω is not zero.

3. Geostrophic Vortices

If we neglect the variation of the Coriolis force with latitude and take

$$\omega \cos \theta = \omega \cos \theta_1 = \omega_1 ,$$

then the last equations of Section 2 reduce to

$$(3.1) \qquad \eta_t = \frac{1}{a \sin \theta} \left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi} \right] ;$$

$$(3.2) \qquad \frac{\partial \tilde{u}}{\partial t} - 2\omega_1 \tilde{v} = - \frac{gh_o}{a \sin \theta} \frac{\partial \eta}{\partial \phi} ;$$

$$(3.3) \qquad \frac{\partial \tilde{v}}{\partial t} + 2\omega_1 \tilde{u} = \frac{gh_o}{a} \frac{\partial \eta}{\partial \theta} .$$

Hereafter we confine ourselves to a study of these equations. As will be explained in the sequel, they lead to what are called geostrophic vortices.

The elimination of η from (3.2) and (3.3) gives

$$(3.4) \qquad \frac{\partial}{\partial t} \left\{ \frac{1}{a \sin \theta} \left[\frac{\partial(\tilde{u} \sin \theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right] \right\} = 2\omega_1 \eta_t .$$

The quantity

$$\zeta = \frac{1}{a \sin \theta} \left[\frac{\partial(\tilde{u} \sin \theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right]$$

is the radial component of vorticity; and by integration of (3.4) we have

$$(3.5) \qquad \frac{1}{a \sin \theta} \left[\frac{\partial(\tilde{u} \sin \theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right] = 2\omega_1 \eta + \zeta(\phi, \theta, 0) - 2\omega_1 \eta(\phi, \theta, 0) .$$

This implies that we can use (3.5), (3.2) and (3.3) as a basic set of equations instead of (3.1); (3.2) and (3.3).

Equations (3.2) and (3.3) imply

$$\frac{\partial^2 \tilde{u}}{\partial t^2} + 4\omega_1^2 \tilde{u} = \frac{gh_o}{a}\left[2\omega_1 \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial^2 \eta}{\partial t \partial \phi}\right] \tag{3.6}$$

and

$$\frac{\partial^2 \tilde{v}}{\partial t^2} + 4\omega_1^2 \tilde{v} = \frac{gh_o}{a}\left[\frac{2\omega_1}{\sin\theta}\frac{\partial \eta}{\partial \phi} + \frac{\partial^2 \eta}{\partial t \partial \theta}\right] . \tag{3.7}$$

If these are used to eliminate $\tilde{u}$ and $\tilde{v}$ from (3.5), we find

$$\begin{aligned} \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\eta}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\eta}{\partial\phi^2} - \frac{a^2}{gh_o}\left[4\omega_1^2\eta + \frac{\partial^2\eta}{\partial t^2}\right] &= \frac{a^2 2\omega_1}{gh_o}\left[\zeta(\phi,\theta,0) - 2\omega_1\eta(\phi,\theta,0)\right] \\ &= \frac{a^2 2\omega_1}{gh_o}\zeta^*(\phi,\theta) . \end{aligned} \tag{3.8}$$

The theory of Laplace transforms can be used to show that the steady state solution of (3.8) is such that

$$L_{t\to\infty}\eta(\phi,\theta,t) = \psi(\phi,\theta) \tag{3.9}$$

where ψ must satisfy

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{4a^2\omega_1^2\psi}{gh_o} = a^2\zeta^*(\phi,\theta) . \tag{3.10}$$

The function ψ does not depend on t and the associated time independent velocity components are

(3.11) $$U = L_{t\to\infty}\tilde{u}(\phi,\theta,t) = \frac{1}{a}\frac{\partial\psi}{\partial\theta} \ ;$$

(3.12) $$V = L_{t\to\infty}\tilde{v}(\phi,\theta,t) = \frac{1}{a\sin\theta}\frac{\partial\psi}{\partial\phi} \ .$$

Note that (3.10) is a consequence of the equations (3.11), (3.12) and the vorticity equation

(3.13) $$\frac{1}{a\sin\theta}\left[\frac{\partial(U\sin\theta)}{\partial\theta} + \frac{\partial V}{\partial\phi}\right] = 2\omega_1 L_{t\to\infty}\eta + \zeta^*(\phi,\theta)$$
$$= \frac{4\omega_1^2\psi}{gh_o} + \zeta^*(\phi,\theta) \ .$$

In other words, (3.10) is implied by (3.5); (3.2) and (3.3) when we ignore inertial forces.

We are interested in the solution of (3.10) when the vorticity $\zeta^*(\phi,\theta)$ is constant in a small neighborhood of a point (ϕ^*,θ^*) and zero elsewhere. This solution, as is well known, can be derived from the solution of the idealized equation which comes from (3.10) when $\zeta^*(\phi,\theta)$ is assumed to represent a vortex of strength μ concentrated at the arbitrary point (ϕ,θ). For this reason we are going to begin with the mathematical interpretation defined by

$$a^2[\zeta(\phi,\theta,0) - 2\omega_1\eta(\phi,\theta,0) = a^2\zeta^*(\phi,\theta) = \frac{a^2\mu\delta(\phi-\phi_1)\delta(\theta-\theta_1)}{a^2\sin\theta_1}$$

where δ symbolizes the Dirac delta function. Hence the basic equation to be solved is

(3.14) $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{4a^2\omega_1^2\psi}{gh_o} = \frac{\mu\delta(\phi-\phi_1)\delta(\theta-\theta_1)}{\sin\theta_1} \ .$$

The motion in a thin planar layer of fluid tangential to the surface of the Earth is often used as an approximation to the actual motion of the Earth's atmosphere in the neighborhood of the point of tangency. For such an approximation the analogues of (3.11); (3.12) and (3.14) are

$$U = \frac{\partial\chi}{\partial r} \; ; \tag{3.15}$$

$$V = \frac{1}{r}\frac{\partial\chi}{\partial\phi} \; ; \tag{3.16}$$

and

$$\frac{1}{r}\left[\frac{\partial rU}{\partial r} + \frac{\partial V}{\partial\phi}\right] = \frac{4\omega_1^2\chi}{gh_o} + \frac{\mu\delta(\phi-\phi_1)\delta(r-r_1)}{r_1} \; . \tag{3.17}$$

In terms of the polar coordinates (r,ϕ), these show that χ must satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\,\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\phi^2} - \frac{4\omega_1^2\chi}{gh_o} = \frac{\mu\delta(\phi-\phi_1)\delta(r-r_1)}{r_1} \; . \tag{3.18}$$

The only physically admissible solution of (3.18) is

$$\chi = -\frac{\mu}{2\pi}\, K_o\left[\frac{2\omega_1}{\sqrt{gh_o}}\sqrt{r^2 + r_1^2 - 2rr_1 \cos\,(\phi-\phi_1)}\right] \tag{3.19}$$

where $K_o[\;\;]$ denotes the zeroth order modified Bessel function of the second kind. This defines what is called a geostrophic vortex. The motion of various configurations of such vortices has been studied by several authors, notably G. K. Morikawa [1], [3] whose papers contain detailed explanations and references. In keeping with what seems to be accepted terminology we can say that (3.14)

defines a geostrophic vortex on a sphere.

Our object now is to investigate the nature of the solution of (3.14) when ω is not zero. This equation can be written as

$$(3.20) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \operatorname{din}\theta \frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \nu(\nu+1)\psi = \frac{\mu\delta(\phi-\phi_1)\delta(\theta-\theta_1)}{\sin\theta_1}$$

where

$$(3.21) \qquad \nu(\nu+1) = -\frac{4\omega_1^2 a^2}{gh_o} .$$

The solution of (3.20) can be inferred from the solution of

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \nu(\nu+1)\psi = 0 .$$

This equation defines the Legendre functions of degree ν. Its general solution can be expressed in the form

$$\psi = c_1 P_\nu(-\cos\theta) + c_2 P_\nu(\cos\theta)$$

when ν is neither zero nor an integer as in the case when ν is defined by (3.21) from which

$$\nu = -\frac{1}{2} + iq$$

where

$$i = \sqrt{-1} ;$$

$$q = \frac{1}{2}\sqrt{\frac{16\omega_1^2 a^2}{gh_o} - 1}$$

and $\omega_1 \neq 0$. In the neighborhood of the north pole $P_\nu(\cos\theta)$ is continuous and $P_\nu(1) = 1$; but in this neighborhood

$$P_\nu(-\cos\theta) = P_{-\frac{1}{2}+iq}(-\cos\theta)$$

$$= \frac{2}{\pi}\cosh q\pi \int_0^\infty \frac{\cos qx\,dx}{\sqrt{2(\cosh x - \cos\theta)}}$$

behaves like

$$P_\nu(-\cos\theta) \sim \frac{2\sin\nu\pi}{\pi}\cdot \ln\theta \ .$$

This suggests that if a vortex is concentrated at (ϕ_1,θ_1); if d_1 is the geodesic distance from this point to an arbitrary point (ϕ,θ) on S; and if σ_1 is the angle $\sigma_1 = d_1/a$, then the fundamental solution of (3.20) is

$$\frac{\mu}{4\sin\nu\pi} P_\nu(-\cos\sigma_1) \equiv \frac{\mu}{4\sin\pi\nu} P_\nu\begin{bmatrix} -\cos\theta\cos\theta_1 \\ -\sin\theta\sin\theta_1\cos(\phi-\phi_1)\end{bmatrix} .$$

It can now be verified by direct substitution that

(3.22) $$\psi = \frac{\mu}{4\sin\nu\pi} P_\nu(-\cos\sigma_1)$$

does indeed satisfy (3.20). Furthermore, computations with the velocity components

$$U = \frac{\mu}{4a\sin\nu\pi}\frac{\partial}{\partial\theta} P_\nu(-\cos\sigma_1) \ ;$$

$$V = \frac{\mu}{4a\sin\theta\sin\nu\pi}\frac{\partial}{\partial\phi} P_\nu(-\cos\sigma_1) \ ;$$

show that in the neighborhood of (ϕ_1,θ_1), (3.22) defines a vortical motion; but the vortex point itself remains at rest — it possesses no autonomous motion. In fact it can be shown that

$$L_{a\to\infty} \frac{\mu}{4 \sin \nu\pi} P_\nu(-\cos \sigma_1)$$

$$= - \frac{\mu}{4\pi} K_o \left[\frac{2\omega_1}{\sqrt{gh_o}} \sqrt{r^2 + r_1^2 - 2rr_1 \cos (\phi-\phi_1)} \right]$$

Hence we can say that (3.22) represents a geostrophic vortex on a rotating sphere in the same way as we say that

$$\chi = - \frac{\mu}{4\pi} K_o \left[\frac{2\omega_1}{\sqrt{gh_o}} \sqrt{r^2 + r_1^2 - 2rr_1 \cos (\phi-\phi_1)} \right]$$

represents a geostrophic vortex on a rotating plane.

For an arbitrary distribution of n+1 vortices on the sphere the function ψ, a stream function is

$$\psi = \frac{1}{4 \sin \nu\pi} \sum_{j=0}^{n} \mu_j P_\nu(-\cos \sigma_j)$$

where $\sigma_j = d_j/a$ and d_j is the geodesic distance on S from the point of concentration of the j-th vortex, (ϕ_j, θ_j), to an arbitrary point (ϕ, θ). That is,

$$\cos \sigma_j = \cos \theta \cos \theta_j + \sin \theta \sin \theta_j \cos (\phi-\phi_j) .$$

The associated velocity field is given by

$$U = \frac{1}{a} \frac{\partial\psi}{\partial\theta} = \frac{1}{4a \sin \nu\pi} \frac{\partial}{\partial\theta} \left\{ \sum_{j=0}^{n} \mu_j P_\nu(-\cos \sigma_j) \right\} ;$$

$$V = \frac{1}{a \sin \theta} \frac{\partial\psi}{\partial\phi} = \frac{1}{4a \sin \theta \sin \nu\pi} \frac{\partial}{\partial\phi} \left\{ \sum_{j=0}^{n} \mu_j P_\nu(-\cos \sigma_j) \right\} .$$

The path of the k-th vortex is along a stream line,

($\psi(x,y)$ = const), and its velocity is equal to the velocity at (ϕ_k, θ_k) due to all of the other vortices. Therefore since

$$U = a \sin\theta \frac{d\phi}{dt} ; \quad V = -a \frac{d\theta}{dt} ;$$

the equations for the motion of (ϕ_k, θ_k) are

$$\frac{d\phi_k}{dt} = \frac{1}{4a^2 \sin\theta \sin\nu\pi} \left. \frac{\partial}{\partial\theta} \left\{ \sum_{\substack{j=0 \\ j\neq k}}^{n} \mu_j P_\nu(-\cos\sigma_j) \right\} \right|_{\substack{\phi=\phi_k \\ \theta=\theta_k}}$$

and

$$\frac{d\theta_k}{dt} = \frac{1}{4a^2 \sin\theta \sin\nu\pi} \left. \frac{\partial}{\partial\phi} \left\{ \sum_{\substack{j=0 \\ j\neq k}}^{n} \mu_j P_\nu(-\cos\sigma_j) \right\} \right|_{\substack{\phi=\phi_k \\ \theta=\theta_k}} .$$

If the motion is confined to a part D of the sphere S, boundary conditions must be imposed; and it is not then possible, in general, to express ψ as a finite sum of terms of the type $\frac{\mu_j}{4 \sin\nu\pi} P_\nu(-\cos\sigma_j)$. When D is a cap bounded by a circle of latitude we have a case which is important for applications to certain meteorological phenomena. The assumption that the Coriolis force is independent of latitude decreases in severity as the latitude of the boundary circle of the cap increases.

4. Geostrophic Vortices in a Cap

Suppose that n+1 concentrated geostrophic vortices exist in the cap which covers the north pole and is bounded by the circle of colatitude $\theta = \lambda$. Suppose that along this boundary circle the velocity to the north is zero. In order to study the motion of the vortices we need the stream function which satisfies

$$(4.1) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\psi}{\partial\theta}(\phi,\theta) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \nu(\nu+1)\psi = \sum_{j=0}^{n}\frac{\mu_j\delta(\phi-\phi_j)\delta(\theta-\theta_j)}{\sin\theta_j}$$

for

$$0 \le \theta,\theta_j < \lambda \; ; \qquad 0 \le \phi,\phi_j \le 2\pi \; ;$$

with the boundary conditions

$$(4.2) \qquad V(\phi,\lambda) = \frac{1}{a}\psi_\phi(\phi,\lambda) = 0$$

or

$$(4.3) \qquad \psi(\phi,\lambda) = c \; .$$

The solution of (4.1) subject to (4.3) can be expressed in the form

$$(4.4) \qquad \psi(\phi,\theta) = bP_\nu(\cos\theta) + \frac{1}{4\sin\nu\pi}\sum_{j=0}^{n}\mu_j P_\nu\left[\begin{array}{c}-\cos\theta\cos\theta_j \\ -\sin\theta\sin\theta_j\cos(\phi-\phi_j)\end{array}\right]$$

$$+ \frac{1}{4\sin\nu\pi}\sum_{j=0}^{n}\mu_j\sum_{m=0}^{\infty}C_\nu(m,\lambda)P_\nu^m(\cos\theta)P_\nu^m(\cos\theta_j)\cos m(\phi-\phi_j)$$

where $P_\nu^m(x)$ is the spherical harmonic of degree ν and order m. This function satisfies

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\nu^m(x) + \left[\nu(\nu+1) - \frac{m^2}{1-x^2}\right]P_\nu^m(x) = 0$$

and is such that

$$P_\nu^m(1) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 . \end{cases}$$

$P_\nu^m(x)$ can also be expressed in terms of $P_\nu(x)$. We have

$$P_\nu^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m P_\nu(x)}{dx^m} . \tag{4.5}$$

The expansion

$$P_\nu[-\cos\theta\cos\theta_j - \sin\theta\sin\theta_j\cos(\phi-\phi_j)]$$

$$= \sum_{m=0}^{\infty}\varepsilon_m P_\nu^m(\cos\theta_j)P_\nu^m(-\cos\theta)\cos m(\phi-\phi_j)$$

where

$$0 \leq \theta_j < \theta < \pi ; \quad \phi, \phi_j \text{ real}$$

and

$$\varepsilon_0 = 1 ,$$

$$\varepsilon_m = (-1)^m \frac{2\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} , \quad m > 0 ,$$

$$P_\nu^0(x) = P_\nu(x) ;$$

shows that the boundary condition (4.3) is satisfied if we take

$$b = \frac{c}{P_\nu(\cos\lambda)}$$

and

$$C_\nu(m,\lambda) = -\varepsilon_m \frac{P_\nu^m(-\cos\lambda)}{P_\nu^m(\cos\lambda)} \quad .$$

It can be shown that the solution (4.4) is unique; and that it is analytic in the cap minus the vortex points.

If instead of (4.3) we impose the boundary condition

$$\psi_\theta(\phi,\lambda) = c \ , \tag{4.6}$$

which means that the velocity to the east is kept constant along the boundary circle of the cap, then (4.4) again provides the solution if we now take

$$b = -\frac{c}{\sin\lambda P_\nu'(\cos\lambda)}$$

and

$$C_\nu(m,\lambda) = \varepsilon_m \frac{P_\nu^{m\,\prime}(-\cos\lambda)}{P_\nu^{m\,\prime}(\cos\lambda)}$$

where $P_\nu^{m\,\prime}(x)$ denotes the derivative of $P_\nu^m(x)$.

The series which appear in (4.4) can be summed if the cap is the northern hemisphere. For this case $\lambda = \pi/2$ and

$$C_\nu(m, \tfrac{\pi}{2}) = \mp\varepsilon_m \quad .$$

Then since

$$\sum_{m=0}^{\infty} \varepsilon_m P_\nu^m(\cos\theta) P_\nu^m(\cos\theta_j) \cos m(\phi-\phi_j) \tag{4.7}$$

$$= P_\nu[\cos\theta \cos\theta_j - \sin\theta \sin\theta_j \cos(\phi-\phi_j)]$$

$$0 \leq \theta, \theta_j < \pi \; ; \qquad \theta + \theta_j < \pi \; ; \qquad \phi, \phi_j \text{ real} \; ;$$

we see that

$$(4.8) \quad \psi(\phi,\theta) = bP_\nu(\cos\theta) + \frac{1}{4\sin\nu\pi} \sum_{j=0}^{n} \mu_j P_\nu \begin{bmatrix} -\cos\theta\cos\theta_j \\ -\sin\theta\sin\theta_j\cos(\phi-\phi_j) \end{bmatrix}$$

$$\mp \frac{1}{4\sin\nu\pi} \sum_{j=0}^{n} \mu_j P_\nu \begin{bmatrix} \cos\theta\cos\theta_j \\ -\sin\theta\sin\theta_j\cos(\phi-\phi_j) \end{bmatrix}$$

where for the boundary condition

$$\psi(\phi, \tfrac{\pi}{2}) = c$$

the upper sign is to be taken with

$$b = \frac{c}{P_\nu(0)} \; ;$$

while for

$$\psi_\theta(\phi, \tfrac{\pi}{2}) = c$$

the lower sign is to be taken with

$$b = -\frac{c}{P_\nu'(0)} \; .$$

Notice that $P_\nu[-\cos\theta\cos\theta_j - \sin\theta\sin\theta_j\cos(\phi-\phi_j)]$ is singular at $(\phi,\theta) = (\phi_j,\theta_j)$ and $P_\nu[\cos\theta\cos\theta_j - \sin\theta\sin\theta_j\cos(\phi-\phi_j)]$ is singular at $(\phi,\theta) = \phi_j, \pi - \theta_j)$. It follows from the form of (4.8) that that solution can be constructed from the fundamental solution

$$\frac{1}{4\sin\nu\pi} P_\nu[-\cos\theta\cos\theta_j - \sin\theta\sin\theta_o\cos(\phi-\phi_j)]$$

by using a process of reflection across $\lambda = \pi/2$. A reflection technique for the construction of the solution for a cap other than a hemisphere is unknown when $\nu \neq 0$.

If $\nu = 0$, that is if $\omega = 0$, then (4.1) reduces to

$$(4.9) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \sum_{j=0}^{n}\frac{\mu_j\delta(\phi-\phi_j)\delta(\theta-\theta_j)}{\sin\theta_j} .$$

This equation possesses the fundamental solution

$$\frac{1}{4\pi}\,\ell n\, a^2\left[\tan^2\frac{\theta}{2} + \tan^2\frac{\theta_j}{2} - 2\tan\frac{\theta}{2}\tan\frac{\theta_j}{2}\cos(\phi-\phi_j)\right]$$

$$= \frac{1}{2\pi}\,\mathrm{Re}\,\ell n\, a\left[\tan\frac{\theta}{2}e^{i\phi} - \tan\frac{\theta_j}{2}e^{i\phi_j}\right] .$$

Let us also remark that if we employ the transformation

$$z = (a\tan\frac{\theta}{2})e^{i\phi} = x + iy$$

then the problem of solving (4.9) subject to $0 \le \theta < \lambda$ and a boundary condition can be transformed into the problem of solving

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \sum_{j=0}^{n}\bar{\mu}_j\delta(x-x_j)\delta(y-y_j)$$

subject to $0 \le \sqrt{x^2+y^2} < R$ and a corresponding boundary condition. See Peters, [4]. The latter equation, as is well known, governs the motion of rectilinear vortices; and for many cases it can be solved by using reflection techniques.

The velocity field defined by the stream function (4.4) is given by

$$U = \sin\theta\ \dot{\phi} = \frac{1}{a}\frac{\partial\psi}{\partial\theta}$$

$$4a^2 \sin\nu\pi\cdot\sin\theta\cdot\dot{\phi} = 4\sin\nu\pi\frac{\partial\psi}{\partial\theta} = -4b\sin\nu\pi\cdot\sin\theta\cdot P_\nu'(\cos\theta)$$

$$+\sum_{j=0}^{n}\mu_j P_\nu'\begin{bmatrix}-\cos\theta\cos\theta_j\\ \\ -\sin\theta\sin\theta_j\cos(\phi-\phi_j)\end{bmatrix}\cdot\begin{bmatrix}\sin\theta\cos\theta_j\\ \\ -\cos\theta\sin\theta_j\cos(\phi-\phi_j)\end{bmatrix}$$

$$-\sin\theta\sum_{j=0}^{n}\mu_j\sum_{m=0}^{\infty}C_\nu(m,\lambda)P_\nu^{m\prime}(\cos\theta)P_\nu^m(\cos\theta_j)\cos m(\phi-\phi_j)$$

and

$$V = -a\ \dot{\theta} = \frac{1}{a\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$-4a^2\sin\nu\pi\cdot\sin\theta\cdot\dot{\theta} = 4\sin\nu\pi\frac{\partial\psi}{\partial\phi}$$

$$=\sum_{j=0}^{n}\mu_j P_\nu'\begin{bmatrix}-\cos\theta\cos\theta_j\\ \\ -\sin\theta\sin\theta_j\cos(\phi-\phi_j)\end{bmatrix}\cdot\sin\theta\sin\theta_j\sin(\phi-\phi_j)$$

$$-\sum_{j=0}^{n}\mu_j\sum_{m=0}^{\infty}C_\nu(m,\lambda)P_\nu^m(\cos\theta)P_\nu^m(\cos\theta_j)m\sin m(\phi-\phi_j)\ .$$

The following equations determine the motion of the vortex at (ϕ_k, θ_k):

$$(4.10)\quad 4a^2 \sin \nu\pi \cdot \sin \theta_k \cdot \dot{\phi}_k = -4b \sin \nu\pi \sin \theta_k P_\nu'(\cos \theta_k)$$

$$+ \sum_{\substack{j=0 \\ j\neq k}}^{n} \mu_j P_\nu' \begin{bmatrix} -\cos \theta_k \cos \theta_j \\ -\sin \theta_k \sin \theta_j \cos (\phi_k - \phi_j) \end{bmatrix} \cdot \begin{bmatrix} \sin \theta_k \cos \theta_j \\ -\cos \theta_k \sin \theta_j \cos (\phi_k - \phi_j) \end{bmatrix}$$

$$- \sin \theta_k \sum_{j=0}^{n} \mu_j \sum_{m=0}^{\infty} C_\nu(m,\lambda) P_\nu^{m\prime}(\cos \theta_k) P_\nu^m(\cos \theta_j) \cos m(\phi_k - \phi_j) ,$$

$$(4.11)\quad -4a^2 \text{s.}\ \nu\pi \cdot \sin \theta_k \cdot \dot{\theta}_k =$$

$$+ \sum_{\substack{j=0 \\ j\neq k}}^{n} \mu_j P_\nu' \begin{bmatrix} -\cos \theta_k \cos \theta_j \\ -\sin \theta_k \sin \theta_j \cos (\phi_k - \phi_j) \end{bmatrix} \cdot \sin \theta_k \sin \theta_j \sin (\phi_k - \phi_j)$$

$$- \sum_{j=0}^{n} \mu_j \sum_{m=0}^{\infty} C_\nu(m,\lambda) P_\nu^m(\cos \theta_k) P_\nu^m(\cos \theta_j) m \sin m(\phi_k - \phi_j) .$$

Suppose now that one vortex of strength μ_o is at the north pole and that n $(n \geq 2)$ others each of strength μ, are symmetrically situated on a circle of latitude within the cap. In other words, suppose that

$$(\phi_o, \theta_o) \equiv (\phi_o, 0) ,$$

$$\mu_o = \mu_o$$

while

$$\left.\begin{aligned} (\phi_j, \theta_j) &\equiv \left((j-1)\frac{2\pi}{n}, \gamma\right) \\ \mu_j &= \mu \end{aligned}\right\} \quad j = 1,2,\ldots,n$$

where

$$0 < \gamma < \lambda \, .$$

For this configuration the velocity to the north of the polar vortex is determined by

$$-4a^2 \sin \nu\pi \cdot \dot{\theta}_o = \mu \sum_{j=1}^{n} P_\nu'(-\cos \gamma) \sin \gamma \cdot \sin (\phi_o - \phi_j)$$

$$- \mu \, L_{\theta_k \to 0} \sum_{j=1}^{n} \sum_{m=0}^{\infty} C_\nu(m,\lambda) \frac{P_\nu^m(\cos \theta_k)}{\sin \theta_k} P_\nu^m(\cos \theta_j) m \sin (\phi_o - \phi_j) \, .$$

Using (4.5) we find

$$-4a^2 \sin \nu\pi \cdot \dot{\theta}_o = \mu \sum_{j=1}^{n} P_\nu'(\cos \gamma) \sin \gamma \sin (\phi_o - \phi_j)$$

$$- \mu \sum_{j=1}^{n} C_\nu(1,\lambda) P_\nu'(1) P_\nu(\cos \gamma) \sin \gamma \sin (\phi_o - \phi_j) \, .$$

From this we see that

$$\dot{\theta}_o = 0$$

because

$$\sum_{j=1}^{n} \sin (\phi_o - \phi_j) = \sum_{j=1}^{n} \sin [\phi_o - (j-1) \tfrac{2\pi}{n}] = 0 \, .$$

The velocity to the north of any one of the vortices on the circle of colatitude γ is given by

$$-4a^2 \sin \nu\pi \cdot \sin \gamma \cdot \dot{\theta}_k$$

$$= \mu \sum_{\substack{j=1 \\ j\neq k}}^{n} P_\nu'[-\cos^2\gamma - \sin^2\gamma \cos (k-j) \tfrac{2\pi}{n}]\sin^2\gamma \sin (k-j) \tfrac{2\pi}{n}$$

$$- \mu \sum_{j=1}^{n} \sum_{m=1}^{\infty} C_\nu(m,\lambda)[P_\nu^m(\cos \gamma)]^2 m \sin m (k-j) \tfrac{2\pi}{n} .$$

In this expression each of the sums with respect to j has the form

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \mathcal{F}[\cos (k-j) \tfrac{2\pi}{n}]\sin (k-j) m \tfrac{2\pi}{n} .$$

This sum is equal to zero and so we find that

$$\dot{\theta}_k = 0 .$$

The velocity to the east of any one of the vortices on the circle $\theta = \gamma$ is given by

$$4a^2 \sin \nu\pi \cdot \dot{\phi}_k = -4b \sin \nu\pi P_\nu'(\cos \gamma)$$

$$+ \mu_o[P_\nu'(-\cos \gamma) - C_\nu(0,\lambda)P_\nu'(\cos \gamma)]$$

$$+ \mu \sum_{j=1}^{n-1} P_\nu'[-\cos^2\gamma - \sin^2\gamma \cos j \tfrac{2\pi}{n}]2 \cos \gamma \sin^2 j \tfrac{\pi}{n}$$

$$- \mu \sum_{j=1}^{n} \sum_{m=0}^{\infty} C_\nu(m,\lambda)P_\nu^{m\,\prime}(\cos \gamma)P_\nu^m(\cos \gamma) \cos jm \tfrac{2\pi}{n}$$

$$\equiv 4a^2 \sin \nu\pi \cdot \Omega .$$

We have now shown that if at time $t = 0$ one vortex of strength μ_o is at the north pole and the others of strength μ occupy the positions

$$(\phi_j, \theta_j) \equiv \left((j-1)\frac{2\pi}{n}, \gamma\right), \quad j = 1,2,\ldots,n ,$$

then the polar vortex stays at the north pole while the others remain on the circle of colatitude $\theta = \gamma$ and each moves along it with constant angular velocity Ω. The strengths μ_o and μ can be chosen so that $\Omega = 0$. On the other hand, if μ_o and μ are prescribed it may be possible (depending on the boundary condition) to choose b and γ so that the vortices are stationary.

5. Linear Stability Equations

The motion of the vortices described in the last section is given by

$$
(5.1) \qquad \begin{aligned}
(\phi_0, \theta_0) &= (\phi_0, 0) \\
(\phi_j, \theta_j) &= ((j-1)\tfrac{2\pi}{n} + \Omega t, \gamma), \quad j = 1, 2, \ldots, n .
\end{aligned}
$$

In order to investigate the stability of this motion consider a second set of vortices with one member of strength μ_0 situated at

$$
(5.2) \qquad (\phi_0, \theta_0) = (\alpha_0 + \Omega t, \beta_0)
$$

and with the others of strength μ situated at

$$
(5.3) \qquad (\phi_j, \theta_j) = (\alpha_j + (j-1)\tfrac{2\pi}{n} + \Omega t, \gamma + \beta_j), \quad j = 1, 2, \ldots, n .
$$

Suppose that at $t = 0$ the quantities α_j, $j = 1, 2, \ldots, n$; and β_j, $j = 0, 1, 2, \ldots, n$, are small. If these quantities remain small as time goes on then the second set of vortices will always be close to the first set (5.1). A condition for linear stability can be found if we can linearize the equations which result when the coordinates of the second set of vortices are substituted in the motion equations (4.10) and (4.11). The linearization cannot be performed directly because α_0 need not be small in order to have the second set of vortices near the first set. However, it follows from our supposition that

$$\left.\begin{array}{l} x_o = \beta_o \cos \alpha_o \\ \qquad\qquad\qquad\quad ; \\ y_o = \beta_o \sin \alpha_o \end{array}\quad \begin{array}{l}\alpha_j \\ \\ \beta_j\end{array}\right\} \quad j = 1,2,\ldots,n$$

are small at $t = 0$; and linearization with respect to these quantities is sufficient for the purpose of developing a linear stability criterion.

After somewhat lengthy computations the linear stability equations turn out to be:

$$(5.4) \qquad q_1(\dot{x}_o - \Omega y_o) = (A_o - B_o \sum_{j=0}^{n-1} \sin^2 j \frac{2\pi}{n}) y_o$$

$$+ C_o(\gamma) \sum_{j=1}^{n} \alpha_j \cos (j-1) \frac{2\pi}{n} + C_o'(\gamma) \sum_{j=1}^{n} \beta_j \sin (j-1) \frac{2\pi}{n} ,$$

$$(5.5) \qquad q_1(\dot{y}_o + \Omega x_o) = -(A_o - B_o \sum_{j=0}^{n-1} \cos^2 j \frac{2\pi}{n}) x_o$$

$$+ C_o(\gamma) \sum_{j=1}^{n} \alpha_j \sin (j-1) \frac{2\pi}{n} - C_o'(\gamma) \sum_{j=1}^{n} \beta_j \cos (j-1) \frac{2\pi}{n} ,$$

$$(5.6) \qquad -q_1 \sin \gamma \cdot \dot{\beta}_k = \frac{\mu_o}{\mu} C_o(\gamma)[x_o \sin (k-1) \frac{2\pi}{n} - y_o \cos (k-1) \frac{2\pi}{n}]$$

$$+ \sum_{\substack{j=1 \\ j \neq k}}^{n} A_{kj}\beta_j + \sum_{\substack{j=1 \\ j \neq k}}^{n} B_{kj}(\alpha_k - \alpha_j) ,$$

$$(5.7) \qquad q_1 \sin \gamma \cdot \dot{\alpha}_k = - \frac{\mu_o}{\mu} C_o'(\gamma)[x_o \cos (k-1) \frac{2\pi}{n} + y_o \sin (k-1) \frac{2\pi}{n}]$$

$$- \sum_{\substack{j=1 \\ j \neq k}}^{n} A_{kj}\alpha_j + \sum_{\substack{j=1 \\ j \neq k}}^{n} C_{kj}\beta_j + C\beta_k$$

where

$$q_1 = \frac{4a^2 \sin \nu\pi}{\mu} ,$$

$$A_o = \frac{4b \sin \nu\pi}{\mu} P_\nu'(1) + \frac{\mu_o}{\mu} P_\nu'(1)[C_\nu(0,\lambda) - C_\nu(1,\lambda)P_\nu'(1))]$$

$$+ n[C_\nu(0,\lambda)P_\nu'(1)P_\nu(\cos\gamma)$$

$$+ 2C_\nu(2,\lambda)P_\nu''(1)P_\nu''(\cos\gamma) - P_\nu'(-\cos\gamma)\cos\gamma] ,$$

$$B_o = \sin^2\gamma[P_\nu''(-\cos\gamma) + 4C_\nu(2,\lambda)P_\nu''(1)P_\nu''(\cos\gamma))] ,$$

$$C_o(\gamma) = P_\nu'(-\cos\gamma)\sin\gamma - C_\nu(1,\lambda)P_\nu'(1)P_\nu'(\cos\gamma)\sin\gamma ,$$

$$A_{kj} = P_\nu'(-\cos\gamma_{kj})\sin\gamma\cos\gamma\sin(k-j)\frac{2\pi}{n}$$

$$+ P_\nu''(-\cos\gamma_{kj})2\sin^3\gamma\cos\gamma\sin^2(k-j)\frac{\pi}{n}\sin(k-j)\frac{2\pi}{n}$$

$$+ \sin\gamma \cdot \sum_{m=1}^{\infty} C_\nu(m,\lambda)P_\nu^m(\cos\gamma)P_\nu^{m\prime}(\cos\gamma)m\sin[m(k-j)\frac{2\pi}{n}] ,$$

$$B_{kj} = P_\nu'(-\cos\gamma_{kj})\sin^2\gamma\cos(k-j)\frac{2\pi}{n}$$

$$+ P_\nu''(-\cos\gamma_{kj})\sin^4\gamma\sin^2(k-j)\frac{2\pi}{n}$$

$$- \sum_{m=1}^{\infty} C_\nu(m,\lambda)[P_\nu^m(\cos\gamma)]^2 m^2\cos[m(k-j)\frac{2\pi}{n}] ,$$

$$C_{kj} = -P_\nu'(-\cos\gamma_{kj})\cdot[\sin^2\gamma + \cos^2\gamma\cos(k-j)\frac{2\pi}{n}]$$

$$+ P_\nu''(-\cos\gamma_{kj})\cdot\sin^2 2\gamma \cdot \sin^4(k-j)\frac{\pi}{n}$$

$$+ \sin^2\gamma \cdot \sum_{m=0}^{\infty} C_\nu(m,\lambda)[P_\nu^{m\prime}(\cos\gamma)]^2\cos[m(k-j)\frac{2\pi}{n}] ,$$

$$C = \frac{4b \sin \nu\pi}{\mu} P_\nu''(\cos \gamma) \sin^2\gamma$$

$$+ \frac{\mu_o}{\mu} \sin^2\gamma [P_\nu''(-\cos \gamma) + C_\nu(0,\lambda) P_\nu''(\cos \gamma)]$$

$$+ \sum_{j=1}^{n-1} \left\{ P_\nu'(-\cos \gamma_j) \cos j \frac{2\pi}{n} + P_\nu''(-\cos \gamma_j) \sin^2 2\gamma \cdot \sin^4 j \frac{\pi}{n} \right\}$$

$$+ \sum_{j=1}^{n} \sum_{m=0}^{\infty} C_\nu(m,\lambda) P_\nu^{m''}(\cos \gamma) P_\nu^m(\cos \gamma) \sin^2\gamma \cos mj \frac{2\pi}{n}$$

$$+ \sum_{m=0}^{\infty} C_\nu(m,\lambda) [P_\nu^{m'}(\cos \gamma)]^2 \sin^2\gamma \ ,$$

$$\cos \gamma_{kj} = \cos^2\gamma + \sin^2\gamma \cdot \cos (k-j) \frac{2\pi}{n} \ ,$$

$$\cos \gamma_j = \cos^2\gamma + \sin^2\gamma \ \cos j \frac{2\pi}{n} \ .$$

The linear system (5.4)-(5.7) has constant coefficients. Therefore the investigation of the stability of the solution of the solution of the system can be started by substituting $x_o = a_o e^{st}$, $\alpha_k = a_k e^{st}$, $y_o = b_o e^{st}$, $\beta_k = b_k e^{st}$ for the variables. This substitution leads to the characteristic determinant which theoretically could be studied in the usual way; but it is clear that the required analysis for the general case would be too complicated to allow the deduction of a practicable formula for linear stability. If a free polar vortex is involved with two or more other vortices it is best to subject the characteristic matrix to numerical analysis. C. Leiva, [2], is doing this and he is analyzing the results for the case in which the cap coincides with the northern hemisphere.

When the polar vortex is either absent or kept fixed, the system (5.4)-(5.7) simplifies so much that the development of a

linear stability formula becomes feasible. If the polar vortex is absent, then $\mu_o = 0$. If the polar vortex is present but constrained to remain at the pole, then $x_o = y_o = 0$. Hence for either of these cases our system reduces to

$$(5.8) \qquad -q_2\dot{\beta}_k = \sum_{\substack{j=1 \\ j\neq k}}^{n} A_{kj}\beta_j + \sum_{\substack{j=1 \\ j\neq k}}^{n} B_{kj}(\alpha_k - \alpha_j) ,$$

$$(5.9) \qquad q_2\dot{\alpha}_k = -\sum_{\substack{j=1 \\ j\neq k}}^{n} A_{kj}\alpha_j + \sum_{\substack{j=1 \\ j\neq k}}^{n} C_{kj}\beta_j + C\beta_k$$

where

$$q_2 = q_1 \sin\gamma = \frac{4a^2 \sin\nu\pi \cdot \sin\gamma}{\mu} .$$

For reasons stated above, let us devote the remaining analysis in this paper to the system (5.8), (5.9) and its implications.

Summations show that (5.8), (5.9) possess two invariants given by

$$(5.10) \qquad \sum_{k=1}^{n} \dot{\beta}_k = 0$$

and

$$(5.11) \qquad q_2 \sum_{k=1}^{n} \dot{\alpha}_k = (C+D) \sum_{k=1}^{n} \beta_k$$

where

$$D = \sum_{\substack{k=1 \\ k\neq j}}^{n} C_{kj}$$

$$= \sum_{j=1}^{n-1} \left\{ \begin{array}{l} -P_\nu'(-\cos\gamma_j)\cdot(\sin^2\gamma + \cos^2\gamma \cos j\,\frac{2\pi}{n}) \\ \quad + P_\nu''(-\cos\gamma_j)\sin^2 2\gamma \sin^4 j\,\frac{\pi}{n} \end{array} \right\}$$

$$+ \sin^2\gamma \sum_{j=1}^{n} \sum_{m=0}^{\infty} C_\nu(m,\lambda)[P_\nu^{m'}(\cos\gamma)]^2 \cos mj\,\frac{2\pi}{n} .$$

From (5.10) and (5.11) we obtain

$$\sum_{k=1}^{n} \beta_k = nq_2\xi_1$$

and

$$\sum_{k=1}^{n} \alpha_k = (C+D)n\xi_1 t + n\xi_2$$

where ξ_1 and ξ_2 are small quantities. If we set

$$\beta_k = \tilde{\beta}_k + q_2\xi_1$$

$$\alpha_k = \tilde{\alpha}_k + (C+D)\xi_1 t + \xi_2$$

we see that

(5.12) $$\sum_{k=1}^{n} \tilde{\beta}_k = 0 ,$$

(5.13) $$\sum_{k=1}^{n} \tilde{\alpha}_k = 0$$

and we find that $\tilde{\beta}_k$ and $\tilde{\alpha}_k$ must again satisfy the same equations as β_k and α_k satisfy, namely the system (5.8), (5.9). In matrix form, we have

(5.14) $$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \tilde{b}_k \\ \tilde{a}_k \end{pmatrix} + q_2 \begin{pmatrix} \dot{\tilde{b}}_k \\ \dot{\tilde{a}}_k \end{pmatrix} = 0 .$$

Now if we take

$$\tilde{\beta}_k = \tilde{b}_k e^{st} \; ; \quad \tilde{\alpha}_k = \tilde{a}_k e^{st}$$

then (5.14) reduces to

$$(5.15) \qquad \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \tilde{b}_k \\ \tilde{a}_k \end{pmatrix} + q_2 s \begin{pmatrix} \tilde{b}_k \\ \tilde{a}_k \end{pmatrix} = 0 .$$

Here the submatrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}$ are right circulant $n \times n$ matrices. It can be shown that the characteristic roots are given by

$$(5.16) \qquad \left.\begin{aligned} q_2 s_1(\ell) &= -A(\ell) + \sqrt{-B(\ell)E(\ell)} \\ q_2 s_2(\ell) &= -A(\ell) - \sqrt{-B(\ell)E(\ell)} \end{aligned}\right\} \quad \ell = 1,2,\ldots,n ;$$

where

$$A(\ell) = \sum_{\substack{j=1 \\ j \neq k}}^{n} A_{kj} e^{\ell(j-k)\frac{2\pi i}{n}}$$

$$= -i \sum_{j=1}^{n-1} \left\{ \begin{aligned} & P_\nu'(-\cos\gamma_j) \sin\gamma \cos\gamma \sin j\frac{2\pi}{n} \\ & + P_\nu''(-\cos\gamma_j)\cdot 2\sin^3\gamma \cos\gamma \sin^2 j\frac{\pi}{n} \sin j\frac{2\pi}{n} \\ & + \sum_{m=1}^{\infty} C_\nu(m,\lambda) P_\nu^m(\cos\gamma) P_\nu^{m\,\prime}(\cos\gamma) \sin\gamma \cdot m \sin jm\frac{2\pi}{n} \end{aligned} \right\} .$$

$$\cdot \sin j\ell\frac{2\pi}{n}$$

which is a pure imaginary; while

$$B(\ell) = \sum_{\substack{j=1 \\ j \neq k}}^{n} B_{kj} \left[1 - e^{\ell(j-k)\frac{2\pi i}{n}} \right]$$

$$= \sum_{j=1}^{n-1} \left(1 - \cos j\ell\frac{2\pi}{n}\right) \left\{ \begin{aligned} & P_\nu'(-\cos\gamma_j) \sin^2\gamma \cos j\frac{2\pi}{n} \\ & + P_\nu''(-\cos\gamma_j) \sin^4\gamma \sin^2 j\frac{2\pi}{n} \\ & + \sum_{m=1}^{\infty} C_\nu(m,\lambda) [P_\nu^m(\cos\gamma)]^2 m^2 \cos jm\frac{2\pi}{n} \end{aligned} \right\}$$

and

$$E(\ell) = C + \sum_{\substack{j=1 \\ j\neq k}}^{n} C_{kj} e^{\ell(j-k)\frac{2\pi}{n}}$$

$$= \frac{4b \sin \nu\pi}{\mu} P_\nu''(\cos\gamma)\sin^2\gamma$$

$$+ \frac{\mu_o}{\mu} \sin^2\gamma[P_\nu''(-\cos\gamma) + C_\nu(0,\lambda)P_\nu''(\cos\gamma)]$$

$$+ \sum_{j=1}^{n-1} \left\{ \begin{array}{l} P_\nu'(-\cos\gamma_j)[\cos j\frac{2\pi}{n} - (\sin^2\gamma + \cos^2\gamma \cos j\frac{2\pi}{n})\cos j\ell\frac{2\pi}{n}] \\ + P_\nu''(-\cos\gamma_j)\sin^2 2\gamma \sin^4 j\frac{\pi}{n}\cdot(1+\cos j\ell\frac{2\pi}{n}) \end{array} \right\}$$

$$+ \sum_{j=1}^{n}\sum_{m=0}^{\infty} C_\nu(m,\lambda)P_\nu^m(\cos\gamma)P_\nu^{m''}(\cos\gamma)\sin^2\gamma\cos jm\frac{2\pi}{n}$$

$$+ \sin^2\gamma \sum_{j=1}^{n}\sum_{m=0}^{\infty} C_\nu(m,\lambda)[P_\nu^{m'}(\cos\gamma)]^2 \cos jm\frac{2\pi}{n}\cos j\ell\frac{2\pi}{n}$$

are real.

Since α_k, β_k are supposed to be small at $t = 0$, the quantities $\tilde{\alpha}_k$, $\tilde{\beta}_k$ must be small at $t = 0$. We define the motion given by (5.1) to be linearly stable if $\tilde{\alpha}_k$, $\tilde{\beta}_k$ remain small as time increases. Then, in order to have linear stability it is necessary that the real parts of the roots (5.16) be less than or equal to zero. Since $A(\ell)$ is a pure imaginary, it follows that

(5.17) $$B(\ell)E(\ell) \geq 0$$

is a necessary condition for linear stability. In other words, each root of the set (5.16) must be a pure imaginary number. With this condition satisfied we still need to consider the possible appearance of multiple roots in the set (5.16). For example if $\ell = n$ then $A(n) = 0$; $B(n) = 0$; and consequently there is at least

a double root present. Presumably this means instability since the multiple zero root indicates solutions of (5.14) which have the form t^{m-1}. However, if the zero root is just a double root it can be shown that solutions $\tilde{\alpha}_k$, $\tilde{\beta}_k$ which in part depend linearly on t are precluded by the invariants (5.12) and (5.13).

We notice of course that

$$\alpha_k = \tilde{\alpha}_k + (C+D)\xi_1 t + \xi_2$$

has a part which depends linearly on t. This is a reflection of the fact that if the vortices (5.1) are slightly displaced, then the angular velocity $\Omega(\gamma)$ will change by a small amount. The quantity (C+D) is equal to

$$C+D = 4a^2 \sin \nu\pi \cdot \sin \gamma \frac{\partial\Omega}{\partial\gamma} .$$

Double roots other than a double zero root may imply instability.

If $\mu_o = 0$ the expectation is that at least one domain of linear stability is the cap:

$$0 \leq \gamma < \lambda_s < \lambda .$$

In fact, if μ_o has the same sign as μ, and the polar vortex is kept fixed, then an investigation of the signs of

$$B(\ell),\ E(\ell), \quad \ell = 1, \ldots, n-1 ;$$

shows that the product $B(\ell)E(\ell)$ is positive (no matter what n is) provided γ is sufficiently small. It is interesting to note that

for the case of a ring of Bessel vortices the linear stability condition is not satisfied for the neighborhood of the origin if $n \geq 8$. The boundary angle $\theta = \lambda_s$ can be approximately determined by computing $B(\ell)$ and $\Xi(\ell)$ for various values of γ. For any given case this would require considerable numerical analysis. As an example of what is involved we turn to the case of three vortices on a circle of latitude.

6. Three Vortices on a Circle of Latitude in a Cap. Polar Vortex Fixed or Absent

Our intention is to devote the rest of this paper to a discussion of the case of three vortices of equal strength μ symmetrically situated on a circle of colatitude $0 \leq \theta < \lambda$. When it is admitted, a polar vortex of strength μ_0 is assumed to be held fixed at the north pole. We are interested in the linear stability of motion of the vortices whose positions at any time t are given by

$$(\phi_0, \theta_0) \equiv (\phi_0, 0) \ ,$$

$$(\phi_0, \theta_j) \equiv ((j-1)\frac{2\pi}{3} + \Omega t, \gamma) \ , \quad j = 1,2,3 \ .$$

Our primary object here is not to present an extended numerical analysis; but rather to record the quantities which need to be evaluated for such an analysis.

The angular velocity of the vortices in the equilibrium position is given by

$$(6.1) \quad \frac{4a^2 \sin \nu\pi}{\mu} \cdot \Omega = \frac{-4b \sin \nu\pi}{\mu} \cdot P_\nu'(\cos \gamma)$$
$$+ \frac{\mu_0}{\mu}\,[P_\nu'(-\cos \gamma) - C_\nu(0,\lambda) P_\nu'(\cos \gamma)]$$
$$+ 3P_\nu'(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma)\cdot \cos \gamma$$
$$- \sum_{m=0}^{\infty} C_\nu(m,\lambda) P_\nu^{m\,\prime}(\cos \gamma) P_\nu^m(\cos \gamma)\cdot(1 + 2\cos m\,\frac{2\pi}{3}) \ .$$

For the present case of three vortices we find

$$B(1) = B(2) = B\ ,$$

$$E(1) = E(2) = E$$

where

$$(6.2)\quad B = -\frac{3}{2} P_\nu(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma)\sin^2\gamma$$

$$+ \frac{9}{4} P''_\nu(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma)\sin^4\gamma$$

$$- 3\sum_{m=1}^{\infty} C_\nu(m,\lambda)[P^m_\nu(\cos\gamma)]^2\, m^2 \cos m\frac{2\pi}{3}$$

and

$$(6.3)\quad E = \frac{4b\sin\nu\pi}{\mu}\cdot P''_\nu(\cos\gamma)\sin^2\gamma$$

$$+ \frac{\mu_o}{\mu}[P''_\nu(-\cos\gamma) + C_\nu(0,\lambda)P''_\nu(\cos\gamma)]\sin^2\gamma$$

$$- \frac{3}{2} P'_\nu(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma)\cos^2\gamma$$

$$+ \frac{9}{16} P''_\nu(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma)\sin^2 2\gamma$$

$$+ \sin^2\gamma\cdot\sum_{m=0}^{\infty} C_\nu(m,\lambda)P^m_\nu(\cos\gamma)P^{m''}_\nu(\cos\gamma)\cdot(1 + 2\cos m\frac{2\pi}{3})$$

$$+ \sin^2\gamma\cdot\sum_{m=0}^{\infty} C_\nu(m,\lambda)[P^{m'}_\nu(\cos\gamma)]^2(1 - \cos m\frac{2\pi}{3})\ .$$

For $\ell = 3$, $B(3) = E(3) = 0$. This means a double zero root in the set (5.16) for three vortices. There are no other multiple roots and hence the linear stability criterion is

$$(6.4)\qquad B(\ell)E(\ell) > 0\ ,\quad \ell = 1,2$$

or

$$\text{(6.5)} \qquad BE > 0 \,.$$

If (6.5) holds then the component periods of the motion about the equilibrium position are given by

$$\frac{s(\ell)}{2\pi i} = \frac{A(\ell) \pm i\sqrt{BE}}{2\pi i q_2} \,, \qquad \ell = 1, 2$$

where

$$\text{(6.6)} \quad iA(1) = \frac{3}{2} P_\nu'(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma) \sin\gamma \cos\gamma$$

$$+ \frac{9}{4} P_\nu''(-\cos^2\gamma + \frac{1}{2}\sin^2\gamma) \sin^3\gamma \cos\gamma$$

$$+ \sqrt{3} \sin\gamma \sum_{m=1}^{\infty} C_\nu(m,\lambda) P_\nu^m(\cos\gamma) P_\nu^{m\prime}(\cos\gamma)\, m \sin m\frac{2\pi}{3}$$

and

$$A(2) = -A(1) \,.$$

Recall that if the boundary condition is

$$\psi(\phi, \lambda) = c$$

then

$$b = \frac{c}{P_\nu(\cos\lambda)} \,,$$

$$C_\nu(0,\lambda) = -\frac{P_\nu(-\cos\lambda)}{P_\nu(\cos\lambda)} \,,$$

$$C_\nu(m,\lambda) = -2(-1)^m \frac{\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} \cdot \frac{P_\nu^m(-\cos\lambda)}{P_\nu^m(\cos\lambda)} \,, \qquad m \neq 0 \,.$$

If the boundary condition is

$$\psi_\theta(\phi, \lambda) = c$$

then

$$b = - \frac{c}{\sin\lambda \, P_\nu^{'}(\cos\lambda)} ,$$

$$C_\nu(0,\lambda) = \frac{P_\nu^{'}(-\cos\lambda)}{P_\nu^{'}(\cos\lambda)} ,$$

$$C_\nu(m,\lambda) = 2(-1)^m \frac{\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} \cdot \frac{P_\nu^{m'}(-\cos\lambda)}{P_\nu^{m'}(\cos\lambda)} , \qquad m \neq 0 .$$

If the cap coincides with the sphere then

$$b = 0 ; \quad C_\nu(m,\pi) = 0 .$$

Some ideas about the effects of imposing a boundary condition for an arbitrary cap can be drawn from a comparison of the motion confined to the northern hemisphere with the free motion on the whole sphere. If the cap is the northern hemisphere then

$$C_\nu(m, \tfrac{\pi}{2}) = \mp \varepsilon_m ,$$

$$\varepsilon_0 = 1 ,$$

$$\varepsilon_m = 2(-1)^m \frac{\Gamma(\nu - m-1)}{\Gamma(\nu + m+1)} , \qquad m > 0$$

and the series which appear in the above expressions can be summed by using (4.7). We find that for $\lambda = \frac{\pi}{2}$:

$$(6.7)\quad \frac{4a^2 \sin \nu\pi}{\mu} \cdot \Omega = \frac{-4b \sin \nu\pi}{\mu} \cdot P_\nu'(\cos \gamma)$$

$$+ \frac{\mu_o}{\mu} [P_\nu'(-\cos \gamma) \pm P_\nu'(\cos \gamma)]$$

$$+ 3P_\nu'(-\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \cos \gamma$$

$$\pm [2P_\nu'(\cos 2\gamma) + P_\nu'(\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma)] \cos \gamma ,$$

$$(6.8)\quad iA(1) = \frac{3}{2} P_\nu'(-\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \sin \gamma \cos \gamma$$

$$+ \frac{9}{4} P_\nu''(-\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \sin^3\gamma \cos \gamma$$

$$\mp [P_\nu'(\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma)$$

$$- \frac{1}{2} P_\nu''(\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \sin^2\gamma] \frac{3}{2} \sin \gamma \cos \gamma ,$$

$$A(2) = -A(1) ,$$

$$(6.9)\quad B = -\frac{3}{2} P_\nu'(-\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \sin^2\gamma$$

$$+ \frac{9}{4} P_\nu''(-\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \sin^4\gamma$$

$$\pm [P_\nu'(\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma)$$

$$- \frac{3}{2} P_\nu''(\cos^2\gamma + \tfrac{1}{2} \sin^2\gamma) \sin^2\gamma] \frac{3}{2} \sin^2\gamma ,$$

$$
\begin{aligned}
(6.10) \qquad E = {} & \frac{4b \sin \nu\pi}{\mu} P''_\nu(\cos \gamma) \sin^2\gamma \\
& + \frac{\mu_o}{\mu} [P''_\nu(-\cos \gamma) \mp P''_\nu(\cos \gamma)] \sin^2\gamma \\
& - \frac{3}{2} P'_\nu(-\cos^2\gamma + \frac{1}{2} \sin^2\gamma) \cos^3\gamma \\
& + \frac{9}{16} P''_\nu(-\cos^2\gamma + \frac{1}{2} \sin^2\gamma) \sin^2 2\gamma \\
& \mp \left\{ \begin{aligned} & P'_\nu(\cos 2\gamma) 2 \sin^2\gamma + P''_\nu(\cos 2\gamma) 2 \sin^2 2\gamma \\ & - P'_\nu(\cos^2\gamma + \frac{1}{2} \sin^2\gamma)(2 - \frac{1}{2} \cos^2\gamma) \\ & + P''_\nu(\cos^2\gamma + \frac{1}{2} \sin^2\gamma) \cdot \frac{1}{16} \sin^2 2\gamma \end{aligned} \right\} .
\end{aligned}
$$

If the boundary condition is

$$\psi(\phi, \tfrac{\pi}{2}) = c$$

the upper sign is to be taken with

$$b = \frac{c}{P_\nu(0)} .$$

If the boundary condition is

$$\psi_\theta(\phi, \tfrac{\pi}{2}) = c$$

the lower sign is to be taken with

$$b = - \frac{c}{P'_\nu(0)} .$$

A numerical analysis for the caps

$$0 \leq \theta < \pi$$

and

$$0 \leq \theta < \frac{\pi}{2}$$

will be presented in a separate report by C. Leiva [2].

For the arbitrary cap

$$0 \leq \theta < \lambda < \frac{\pi}{2}$$

each series which appears in the expressions (6.1), (6.2) and (6.6) can be approximated by the sum of its first n terms. Various methods can be used to estimate the error involved. One method is to use the formula

$$(6.11) \qquad (-1)^m \frac{\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} \cdot P_\nu^m(x) = \frac{(1-x^2)^{-m/2}}{\Gamma(m)} \int_x^1 (t-x)^{m-1} P_\nu(t)dt$$

$$= \frac{P_\nu(\bar{t})}{\Gamma(m+1)} \cdot \left(\frac{1-x}{1+x}\right)^{m/2} ,$$

$$x < \bar{t} < 1 \; ; \quad m \geq 1$$

and the formula

$$(6.12) \qquad (-1)^m \frac{\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} [-P_\nu^{m'}(x)] = \frac{P_\nu(\bar{t})}{\Gamma(m)} \cdot \left(\frac{1-x}{1+x}\right)^{m/2} \cdot \frac{1}{1-x^2} ,$$

$$m \geq 2$$

in conjunction with

$$(6.13) \qquad (-1)^m \frac{\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} = - \frac{\cosh q\pi}{\pi P_\nu^m(0) P_\nu^{m'}(0)} .$$

For example, consider the series which appears in the expression

for Ω, namely

$$\sum = - \sum_{m=0}^{\infty} C_\nu(m,\lambda) P_\nu^{m\prime}(\cos\gamma) P_\nu^m(\cos\gamma)(1 + 2\cos m \frac{2\pi}{3}) .$$

Let the boundary condition be

$$\psi(\phi,\lambda) = c$$

so that

$$C_\nu(m,\lambda) = \begin{cases} -\dfrac{P_\nu(-\cos\lambda)}{P_\nu(\cos\lambda)} \\ \\ -2(-1)^m \dfrac{\Gamma(\nu - m+1)}{\Gamma(\nu + m+1)} \dfrac{P_\nu^m(-\cos\lambda)}{P_\nu^m(\cos\lambda)} , & m \neq 0 . \end{cases}$$

If S_2, the sum of the first two terms, is used to approximate $\sum$ we have

$$S_2 - \sum = \frac{6\cosh q\pi}{\pi} \sum_{m=2}^{\infty} \frac{P_\nu^{3m}(-\cos\lambda)}{P_\nu^{3m}(\cos\lambda)} \frac{P_\nu^{3m\prime}(\cos\gamma) P_\nu^{3m}(\cos\gamma)}{P_\nu^{3m\prime}(0) P_\nu^{3m}(0)} .$$

Then from the formulas (6.11) and (6.12) we find

$$\frac{1}{P_\nu(\cos\lambda)[P_\nu(0)]^2} \sum_{m=2}^{\infty} (\cot^2\frac{\lambda}{2}\tan^2\frac{\gamma}{2})^{3m}$$

$$< \frac{\pi(S_2 - \sum)}{6\cosh q\pi} < P_\nu(-\cos\lambda)[P_\nu(\cos\gamma)]^2 \sum_{m=2}^{\infty} (\cot^2\frac{\lambda}{2}\tan^2\frac{\gamma}{2})^{3m} .$$

Since

$$\zeta = \cot\frac{\lambda}{2}\tan\frac{\gamma}{2} < 1$$

we obtain

$$\frac{\zeta^{12}}{P_\nu(\cos\lambda)[P_\nu(0)]^2}\cdot\frac{1}{1-\zeta^6} < \frac{\pi(S_2-\sum)}{6\cosh q\pi} < \frac{P_\nu(-\cos\lambda)[P_\nu(\cos\gamma)]^2\zeta^{12}}{1-\zeta^6} .$$

This shows that $\sum \to \infty$ as $\zeta \to 1$, which means that Ω is large when the vortices on the latitude circle are close to the boundary circle. It also shows that

$$S_2 - \sum \to 0$$

as $\gamma \to 0$. A similar analysis can be applied to the infinite sums in (6.2), (6.3) and (6.6).

References

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[2] Leiva, C., Courant Institute of Mathematical Sciences, New York University. Report to be published.

[3] Morikawa, G. K., and Swenson, E. V., Interacting Motion of Rectilinear Geostrophic Vortices, the Physics of Fluids, Vol. 14, No. 6, June 1971. pp. 1058-1073.

[4] Peters, A. S., Motion of a Vortex in a Rotating Layer of Fluid Between Rigid Concentric Spheres, Courant Institute of Mathematical Sciences, New York University, IMM 398, September 1973.